Homework 5
MTH 869 Algebraic Topology

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Lemma 0.1 (for Exercise 1.3.5, part one). *Let $f : X \to Y$ be a continuous bijection, and suppose that X is compact and Y is Hausdorff. Then f has a continuous inverse. (As a consequence, f is a homeomorphism.)*

Proof. Let $g = f^{-1}$. Suppose $V \subset X$ is closed. Then V is compact since X is compact. Then $f(V)$ is compact since f is continuous. Since Y is Hausdorff, $f(V)$ is closed. Then since $f(V) = g^{-1}(V)$, $g^{-1}(V)$ is closed. Hence the preimage of any closed set under g is closed, so g is continuous. □

Proposition 0.2 (Exercise 1.3.5, part one). *Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ inside the square. For every covering space $p : \widetilde{X} \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\widetilde{X}$.*

Proof. Let S be the set of $y \in [0,1]$ such that there exists a neighborhood of $\{0\} \times [0,y]$ in X that lifts homeomorphically to $\widetilde{X}$. Note that S is not empty, because the point $(0,0)$ has an evenly covered neighborhood, which has a homeomorphic lift. Thus we can define $y_0 = \sup S$. If we can show that $1 \in S$, then by definition of S there is an open neighborhood of the left edge $\{0\} \times [0,1]$ that has a homeomorphic lift.

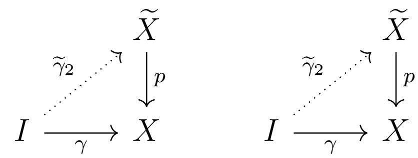
Suppose that $y_0 < 1$, and let U be an open neighborhood of $\{0\} \times [0, y_0]$ with homeomorphic lift $\widetilde{U} \subset \widetilde{X}$, and let $\widetilde{y}_0$ be the unique preimage of $(0, y_0)$ in $\widetilde{U}$. Let V be an evenly covered neighborhood of $(0, y_0)$, so $p^{-1}(V) = \bigsqcup_{i \in I} \widetilde{V}_i$ where each $\widetilde{V}_i$ maps homeomorphically to V under p. Choose the unique i so that $\widetilde{y}_0 \in \widetilde{V}_i$. We claim that there is a homeomorphic lift of a neighborhood of $[0, y]$ where $y > y_0$, where this lift is contained in $\widetilde{U} \cup \widetilde{V}_i$.

Since $p(\widetilde{U}) = U$ and $p(\widetilde{V}_i) = V$, it is immediate that $p|_{\widetilde{U} \cup \widetilde{V}_i} : \widetilde{U} \cup \widetilde{V}_i \to U \cup V$ is surjective. Of course, any restriction of p is continuous. If we can show that this restriction of p is injective, then it is a homeomorphism by the lemma above. To show that $p|_{\widetilde{U} \cup \widetilde{V}_i}$ is injective, we need to show that for $a, b \in \widetilde{U} \cup \widetilde{V}_i$ we have the implication

$$p(a) = p(b) \implies a = b$$

Since $p|_{\widetilde{U}}$ and $p|_{\widetilde{V}_i}$ are injective, if a, b are both in $\widetilde{U}$ or both in $\widetilde{V}_i$, the above implication holds. So we just need to consider the case where $a \in \widetilde{U} \setminus \widetilde{V}_i$ and $b \in \widetilde{V}_i \setminus \widetilde{U}$.

First, suppose that $p(a) = p(b) = (0, y)$, that is, the image is in the left edge. Let $\gamma : I \to X$ be the straight line path from $(0,0)$ to $(0,y)$. Then there exist path lifts $\widetilde{\gamma}_1 : I \to \widetilde{X}$ and $\widetilde{\gamma}_2 : I \to \widetilde{X}$ such that $\widetilde{\gamma}_1(0) = \widetilde{\gamma}_2(0) = p|_{\widetilde{U}}^{-1}(0,0)$, and $\widetilde{\gamma}_1(1) = a$ and $\widetilde{\gamma}_2(1) = b$.



Then by the uniqueness of path lifting (Proposition 1.30 in Hatcher), this implies that $a = b$. Thus $p|_{\widetilde{U} \cup \widetilde{V}_i}$ is injective on the left edge.

Now we show that $p|_{\widetilde{U} \cup \widetilde{V}_i}$ is injective on the rest of $\widetilde{U} \cup \widetilde{V}_i$ by contradiction. For $\epsilon > 0$, define

$$U_\epsilon = U \cap ([0, \epsilon] \times [0, 1]) \qquad \widetilde{U}_\epsilon = p|_{\widetilde{U}}^{-1}(U_\epsilon)$$
$$V_\epsilon = V \cap ([0, \epsilon] \times [0, 1]) \qquad \widetilde{V}_{i,\epsilon} = p_{\widetilde{V}_i}^{-1}(V_\epsilon)$$

If for any $\epsilon > 0$, p is injective on $U_\epsilon \cup V_\epsilon$, then $p|_{\widetilde{U}_\epsilon \cup \widetilde{V}_{i,\epsilon}}$ is injective and hence $\widetilde{U}_\epsilon \cup \widetilde{V}_{i,\epsilon}$ is a homeomorphic lift of $U_\epsilon \cup V_\epsilon$ and we're done. So suppose otherwise. Then for every $n \in \mathbb{N}$, there exist $a_n \in \widetilde{U}_{\frac{1}{n}} \setminus \widetilde{V}_{i,\frac{1}{n}}$ and $b_n \in \widetilde{V}_{i,\frac{1}{n}} \setminus \widetilde{U}_{\frac{1}{n}}$ so that $p(a_n) = p(b_n)$ and $a_n \neq b_n$. Then the sequences a_n and b_n must have convergent subsequences a_{n_k} and b_{n_k} with limits

$$a = \lim_{k \to \infty} a_{n_k} \qquad b = \lim_{k \to \infty} b_{n_k}$$

Since $p(a_n)$ approaches the left edge, $p(a)$ and $p(b)$ are on the left edge. By continuity of p, we have $p(a) = p(b)$. However, $a_n \in \widetilde{U} \setminus \widetilde{V}_i$ implies that $a \in U \setminus V$. Likewise, $b_n \in \widetilde{V}_i \setminus \widetilde{U}$ implies that $b \in V \setminus U$. Thus we have a, b so that $p(a) = p(b)$ on the left edge but $a \neq b$. Since we already showed that $p|_{\widetilde{U} \cup \widetilde{V}_i}$ is injective on the left edge, this implies is a contradiction, so we conclude that $p|_{\widetilde{U} \cup \widetilde{V}_i}$ is injective.

Now that we have shown that $p|_{\widetilde{U} \cup \widetilde{V}_i}$ is injective, it is a homeomorphism onto $U \cup V$. Since V is an open neighborhood of $(0, y_0)$ and $y_0 < 1$, V contains some point $(0, y)$ with $y > y_0$. Thus $U \cup V$ is a neighborhood of $[0, y]$ that lifts homeomorphically to $\widetilde{X}$. This contradicts the fact that y_0 is the supremum of S, so we conclude that $\sup S$ cannot be less than one. Since S is a subset of $[0, 1]$, this means that $\sup S = 1$. Thus by definition of S, there is an open neighborhood of the left edge that lifts homeomorphically to $\widetilde{X}$. $\square$

Proposition 0.3 (Exercise 1.3.5, part two)**.** *Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ inside the square. Then X has no simply connected covering space.*

Proof. Let $p : \widetilde{X} \to X$ be a covering map. Using part one, let A be an open neighborhood of the left edge with a homeomorphic lift $\widetilde{A} \subset \widetilde{X}$.

We claim that there exists $K \in \mathbb{N}$ so that $\left[0, \frac{1}{K}\right] \times [0,1] \subset A$. Suppose there is no such K. Then for every $k \in \mathbb{N}$, there exists $\left(\frac{1}{k}, y_k\right) \in X \setminus A$. Since this sequence is bounded, it has a convergent subsequence, which must converge to a point on the left edge $\{0\} \times [0,1]$. But A is open, so $X \setminus A$ is closed, so $X \setminus A$ contains its limit points. This contradicts the fact that A covers the left edge. Thus the claimed K exists. This implies that A contains the nontrivial rectangular loop

$$\gamma = \Big(\{0\} \times [0,1]\Big) \cup \left(\left\{\frac{1}{K+1}\right\} \times [0,1]\right) \cup \left(\left[0, \frac{1}{K+1}\right] \times \{0\}\right) \cup \left(\left[0, \frac{1}{K+1}\right] \times \{1\}\right)$$

which lifts to a nontrivial loop $\widetilde{\gamma}$ in $\widetilde{A}$. Then thinking of $\widetilde{\gamma}$ as a loop in the larger space $\widetilde{X}$, we see that $p_*[\widetilde{\gamma}] = [\widetilde{\gamma} \circ p] = [\gamma]$, so the image of $\pi_1(\widetilde{X})$ is nontrivial in $\pi_1(X)$. This implies that $\pi_1(\widetilde{X})$ cannot be trivial, so $\widetilde{X}$ is not simply connected. □

Proposition 0.4 (Exercise 1.3.7)**.** *Let Y be the quasi-circle consisting of the closed subspace of $\mathbb{R}^2$ given by a piece of the graph of $y = \sin(1/x)$ and the segment $[-1,1]$ with an arc connecting the two pieces. Induce a map $f : Y \to S^1$ by collapsing the segment $[-1,1]$ Then f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Consequently, local path-connectedness of Y is a necessary hypothesis for the lifting criterion.*

Proof. We think of Y as the union of three pieces $Y = A \cup B \cup C$, where A is the vertical segment $[-1,1]$, B is connecting arc, and C is the piece of the graph of $\sin(1/x)$. We have a quotient map $q : Y \to Y/A$ by collapsing A to a point, which we denote by a. Then we have a map $g : Y/A \to S^1$ by leaving a and B fixed, and projecting down the graph of $\sin(1/x)$ to the x-axis. By doing a rotation if necessary, we can assume that $g(a) = 1$ (we are thinking of S^1 as the unit circle in $\mathbb{C}$). Then the map f is the composition $f = g \circ q$.

Let $p : \mathbb{R} \to S^1$ be the usual covering map $t \mapsto e^{it}$, and suppose there is a lift $\widetilde{f} : Y \to \mathbb{R}$.

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow p \\ Y & \xrightarrow{f} & S^1 \end{array}$$

For $\epsilon > 0$, define

$$U_\epsilon = \{y \in Y : \text{dist}(y, A) > \epsilon\}$$
$$V_\epsilon = \{y \in Y : \text{dist}(y, A) < \epsilon\}$$

That is, U_ϵ is an open subset of Y covering almost all of Y, except for avoiding a ϵ-neighborhood of A, and V is an ϵ-neighborhood of A. Note that $U_\epsilon \cup V_{2\epsilon} = Y$. Then $f(U_\epsilon)$ covers almost all of S^1, and $f(V_{\epsilon/2})$ is a small neighborhood of 1.

$$p\widetilde{f}(U_\epsilon) = f(U_\epsilon) = \{x \in S^1 : \text{dist}(x, 1) > \epsilon\}$$
$$p\widetilde{f}(V_{2\epsilon}) = f(V_{2\epsilon}) = \{x \in S^1 : \text{dist}(x, 1) < 2\epsilon\}$$

This says that $\widetilde{f}(U_\epsilon)$ is an interval of length just a bit smaller than 2π that avoids $f(a) = 2\pi k$, and $\widetilde{f}(V_\epsilon)$ is an small interval containing $f(a) = 2\pi k$. Since U_ϵ and $V_{2\epsilon}$ over lap, the images overlap. Thus the union

$$\widetilde{f}(Y) = \widetilde{f}(U_\epsilon) \cup \widetilde{f}(V_{2\epsilon})$$

is a single open interval of length greater than 2π. So there exist $\alpha, \beta \in \widetilde{f}(Y)$ such that $|\alpha - \beta| = 2\pi$. Since $\widetilde{f}(U_\epsilon)$ and $\widetilde{f}(V_{2\epsilon})$ are both intervals of length less than 2π, α and β can't be in the same one. WLOG assume $\alpha \in \widetilde{f}(U_\epsilon) \setminus \widetilde{f}(V_{2\epsilon})$ and $\beta \in \widetilde{f}(V_{2\epsilon}) \setminus \widetilde{f}(U_\epsilon)$. Then there exist $y_\alpha \in U_\epsilon \setminus V_{2\epsilon}$ and $y_\beta \in V_{2\epsilon} \setminus U_\epsilon$ with $\widetilde{f}(y_\alpha) = \alpha$ and $\widetilde{f}(y_\beta) = \beta$. Then

$$|\alpha - \beta| = 2\pi \implies p(\alpha) = p(\beta) \implies p\widetilde{f}(y_\alpha) = p\widetilde{f}(y_\beta) \implies f(y_\alpha) = f(y_\beta)$$

By construction of U_ϵ, $y_\alpha \in U_\epsilon$ implies that y_α is not in A. Since f is injective except for values in A, this implies that $y_\alpha = y_\beta$, which contradicts the fact that y_α and y_β lie in disjoint neighborhoods of Y. Therefore, no lift $\widetilde{f}$ exists. □

Proposition 0.5 (Exercise 1.3.8). *Let $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to Y$ be simply connected covering spaces of the path connected, locally path conneced spaces X and Y respectively. If $X \simeq Y$, then $\widetilde{X} \simeq \widetilde{Y}$.*

Proof. Let $f : Y \to X$ and $g : Y \to X$ so that $fg \simeq \mathrm{Id}_Y$ and $gf \simeq \mathrm{Id}_X$.

$$\begin{array}{ccc} \widetilde{X} & & \widetilde{Y} \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ X & \underset{g}{\overset{f}{\rightleftarrows}} & Y \end{array}$$

Since $\widetilde{X}$ and $\widetilde{Y}$ are simply connected, the induced maps $(p_X)_*$ and $(p_Y)_*$ are trivial. Then by Proposition 1.33, there exist lifts $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ and $\widetilde{g} : \widetilde{Y} \to \widetilde{X}$ so that the following diagrams commute.

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{\widetilde{g}}{\nearrow} & \downarrow{\scriptstyle p_X} \\ \widetilde{Y} & \xrightarrow[g p_Y]{} & X \end{array} \qquad \begin{array}{ccc} & & \widetilde{Y} \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow{\scriptstyle p_Y} \\ \widetilde{X} & \xrightarrow[f p_X]{} & Y \end{array}$$

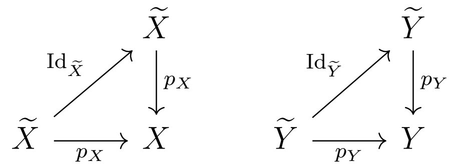
Then notice that

$$p_X \widetilde{g}\widetilde{f} = g p_Y \widetilde{f} = g f p_X$$
$$p_Y \widetilde{f}\widetilde{g} = f p_X \widetilde{g} = f g p_Y$$

that is, the following diagrams commute:

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{\widetilde{g}\widetilde{f}}{\nearrow} & \downarrow{\scriptstyle p_X} \\ \widetilde{X} & \xrightarrow[g f p_X]{} & X \end{array} \qquad \begin{array}{ccc} & & \widetilde{Y} \\ & \overset{\widetilde{f}\widetilde{g}}{\nearrow} & \downarrow{\scriptstyle p_Y} \\ \widetilde{Y} & \xrightarrow[f g p_Y]{} & Y \end{array}$$

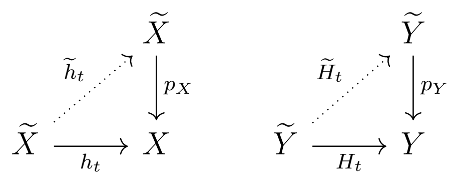
We have the obvious lifts



Note that

$$
\begin{aligned}
gf \simeq \mathrm{Id}_X &\implies gfp_X \simeq p_X \\
fg \simeq \mathrm{Id}_Y &\implies fgp_Y \simeq p_Y
\end{aligned}
$$

So we have homotopies $h_t : \widetilde{X} \to X$ and $H_t : \widetilde{Y} \to Y$ with $h_0 = p_X, h_1 = gfp_X$ and $H_0 = p_Y, H_1 = fgp_Y$. We have a lift $\widetilde{h}_0 = \mathrm{Id}_{\widetilde{X}}$ of h_0 and a lift $\mathrm{Id}_{\widetilde{Y}}$ of $\widetilde{H}_0$, so by Proposition 1.30 we have homotopy lifts $\widetilde{h}_t$ and $\widetilde{H}_t$ making the following diagrams commute.



That is, $p_X\widetilde{h}_1 = h_1 = gfp_X$ and $p_Y\widetilde{H}_1 = H_1 = fgp_Y$. So we have two lifts of gfp_X, namely $\widetilde{h}_1$ and $\widetilde{g}\widetilde{f}$. Likewise, we have two lifts of fgp_Y, which are $\widetilde{H}_1$ and $\widetilde{f}\widetilde{g}$. By uniqueness of the homotopy lifts, this implies that $\widetilde{h}_1 = \widetilde{g}\widetilde{f}$ and $\widetilde{H}_1 = \widetilde{f}\widetilde{g}$. Hence $\widetilde{g}\widetilde{f} \simeq \mathrm{Id}_{\widetilde{X}}$ and $\widetilde{f}\widetilde{g} \simeq \mathrm{Id}_{\widetilde{Y}}$, so $\widetilde{X} \simeq \widetilde{Y}$ via the homotopy equivalence $\widetilde{f}$. □

Lemma 0.6 (for Exercise (b))**.** *There is no injective group homomorphism $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$.*

Proof. Suppose $\phi : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is an injective group homomorphism. Let $\mathbb{Z} * \mathbb{Z}$ be generated by x, y. Then since $\mathbb{Z} \times \mathbb{Z}$ is abelian,

$$\phi(yx) = \phi(y)\phi(x) = \phi(x)\phi(y) = \phi(xy)$$

Because ϕ is injective, this implies $xy = yx$, which is not true in $\mathbb{Z} * \mathbb{Z}$. □

Proposition 0.7 (Exercises (a),(b),(c))**.** *Consider the following four spaces: $S^1 \times S^1, S^2$, $S^1 \times \mathbb{R}$, and $\mathbb{R}^2 \setminus \{x, y\}$ where x, y are two distinct points in $\mathbb{R}^2$. Then*

- *a) $S^1 \times S^1$ is not a covering space of any of the other three.*
- *b) $\mathbb{R}^2 \setminus \{x, y\}$ is not a covering space of any of the other three.*
- *c) S^2 cannot be a covering space of any of the other three.*

Proof. First note that $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$, $\pi_1(S^1 \times \mathbb{R}) \cong \mathbb{Z}, \pi_1(S^2) \cong 0$, and $\pi_1(\mathbb{R}^2 \setminus \{x, y\} \cong \mathbb{Z} * \mathbb{Z}$. Also, the universal cover of $S^1 \times S^1$ is $\mathbb{R} \times \mathbb{R}$, the universal cover of S^2 is itself, and the universal cover of $S^1 \times \mathbb{R}$ is $\mathbb{R} \times \mathbb{R}$.

(a) A covering map p from $S^1 \times S^1$ to S^2 or $S^1 \times \mathbb{R}$ would induce an injective homomorphism $\mathbb{Z} \times \mathbb{Z} \to 0$ or $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ respectively, neither of which is possible. A covering map from $S^1 \times S^1$ to $\mathbb{R} \setminus \{x, y\}$ would induce an injective group homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$. If such a homomorphism existed, then $\mathbb{Z} * \mathbb{Z}$ would have a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. However, every subgroup of a free group is free, so this is a contradiction.

(b) A covering map from $\mathbb{R}^2 \setminus \{x, y\}$ would induce an injective homomorphism from $\mathbb{Z} * \mathbb{Z}$, but this group does not inject into any of $\mathbb{Z} \times \mathbb{Z}, 0$, or $\mathbb{Z}$, so $\mathbb{R}^2 \setminus \{x, y\}$ cannot be a covering space of any of the other three.

(c) If S^2 is a covering space of any of the other three, then since S^2 is simply connected it is isomorphic as a cover to the universal cover. Since S^2 is not homeomorphic to $\mathbb{R}^2$, the universal cover for $S^1 \times S^1$ and $S^1 \times \mathbb{R}$, S^2 is not a cover of either of the those. S^2 cannot be a covering space of $\mathbb{R}^2 \setminus \{x, y\}$ because the image of a continuous map from S^2 must be comact, so there is no continuous surjective map $S^2 \to \mathbb{R}^2 \setminus \{x, y\}$. $\square$